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# THE EXAMPLES OF NONLINEAR INSTABILITY IN HYDRODYNAMICS.

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## Introduction.

In this paper we consider well-known equilibrium states of ideal fluid. There is a free boundary in every case. This is either a free surface of a fluid volume or a interface of two fluids. The form of the free surface determines the value of potential energy. There are infinitesimal perturbations providing a negative value of the potential energy ( if the potential energy of an equilibrium state is assumed equal to zero ). That is the potential energy does not have a strict minimum in the positions of equilibrium and an instability is expected. Indeed, the instability of the motions in question is not striking from physical standpoint. However it has a little mathematical basis under nonlinear formulation.

For example, one of the well-known unstable states is so-called "a rest fluid at the ceiling": at the initial time a rest fluid occupies only the top part of a cylindrical container and the gravity force has natural direction. Physical situation is very simple: this state of any fluid is impossible. But investigation of the stability for a mathematical model is a rather difficult problem.

We have proved the increase of perturbations which are solutions of the exact nonlinear equations of motion. Therefore we say about the examples of nonlinear instability.

We use some analogy of the Lyapunov direct method. The examples are connected by the form of the Lyapunov functional. The functional used in the stability theory of mechanical systems [2] and the known definition of the canonical variables in hydrodynamics ( see [5]. pp. 33-34) heuristically determine this form.

In section 1 we consider the hydrostatic equilibrium of two immiscible fluids in a cylinder when the heavier fluid occupies the top part. We study characteristics of a motion caused by a perturbation of the interface. The growth of a deformation of the interface is proved.

The development of perturbations of a solid body rotation of an ideal fluid is studied in section 2. The movement takes place in weightlessness. However, in plane case, we can define the potential energy of the motion in a rotating system and can also use the Lyapunov direct method.

## 1. The motion of two-layer fluid in cylindrical containers.

We consider the problem on a simultaneous motion of two immiscible incompressible ideal fluids in a cylinder $\Omega$. The cylinder has a generator parallel to z-axis ( in the Cartesian coordinates $x$ , $y$ , $z$ ) and end-walls given by $z = H_1$ and $z = -H_2$ ( $H_i > 0$, $i = 1, 2$). A section of the cylinder with a plane $z = const$ has the smooth boundary. The cylindrical container is placed in the field of gravity $\mathbf{g} = (0, 0, -g)$, $g > 0$.

The first fluid has the density $\rho_1$ and occupies the top part $\Omega_1^t$ of the cylinder. The second fluid with the density $\rho_2$ occupies the bottom part $\Omega_2^t$. The fluids are separated with an interface $\Gamma^t$ given by

$$z = \eta(x, y, t), \qquad t \geq 0.$$

Let $\mathit{\Gamma}_0 = \Omega \cap \{z = 0\}$ is the boundary between fluids in equilibrium state.

The surface tension on $\mathit{\Gamma}^t$ is taken into account. All quantities are dimensionless. The jumps of functions on the interface $\mathit{\Gamma}^t$ are marked by the square brackets: $[\rho] = \rho_1 - \rho_2$, etc.

A motion of both fluids is described by the Euler equations:

$$\frac{\partial \mathbf{u}_\lambda}{\partial t} + (\mathbf{u}_\lambda \cdot \nabla)\, \mathbf{u}_\lambda = -\frac{1}{\rho_\lambda} \nabla p_\lambda + \mathbf{g}, \tag{1.1}$$

$$\frac{\partial u_\lambda}{\partial x} + \frac{\partial v_\lambda}{\partial y} + \frac{\partial w_\lambda}{\partial z} = 0 \qquad \text{in } D_\lambda = \big\{(\mathbf{x}, t) : \mathbf{x} = (x, y, z) \in \Omega_\lambda^t, t > 0\big\}; \tag{1.2}$$

Here $\mathbf{u}_\lambda = (u_\lambda, v_\lambda, w_\lambda)$ is the velocity of the fluid ; $p_\lambda$ is the pressure.

The boundary conditions have a well-known form:
the impenetrability condition on the rigid walls

$$\mathbf{u}_\lambda \cdot \mathbf{n} = 0 \qquad \text{on } \partial\Omega; \tag{1.3}$$

the kinematic and dynamic conditions on the free interface

$$\frac{\partial \eta}{\partial t} + u_\lambda \frac{\partial \eta}{\partial x} + v_\lambda \frac{\partial \eta}{\partial y} = w_\lambda, \tag{1.4}$$

$$[p] + \alpha \nabla_2 \cdot \left( \frac{\nabla_2 \eta}{\sqrt{1 + |\nabla_2 \eta|^2}} \right) = 0 \qquad \text{on } \Gamma^t, \tag{1.5}$$

where $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$; $\alpha$ is a coefficient of surface tension; $\nabla_2 = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$.

If we are interested in a particular solution we should set initial data:

$$\mathbf{u}_\lambda(x, y, z, 0) = \mathbf{u}_\lambda^0(x, y, z), \qquad \eta(x, y, 0) = \eta_0(x, y) \qquad \left( \int\limits_{\Gamma_0} \eta_0(x, y)\, dxdy = 0 \right). \tag{1.6}$$

Any solution of the initial-boundary value problem (1.1)-(1.6) describes some perturbation of the hydrostatic equilibrium

$$\mathbf{u}_\lambda \equiv 0, \qquad \eta \equiv 0, \qquad p_\lambda = -g\rho_\lambda z + p_0. \tag{1.7}$$

It is quite clear, that functions (1.7) give an elementary steady solution of the problem.

Our goal is to prove the instability of the hydrostatic equilibrium (1.7). We say about the instability because we consider the only case where the heavier fluid occupies the top part of the container, that is

$$\rho_1 > \rho_2,$$

and we have a case of the Rayleigh-Taylor instability.

Unfortunately, we do not know any results related to the solvability of the problem (1.1)-(1.6). The main obstacles are connected with the nonlinearity, the presence of the free interface and the presence of the contact between the rigid boundary and the free one. However, the solvability is not so important because we are interested in an instability. We just suppose that the problem (1.1)-(1.6) has a smooth solution over the infinite time interval for any initial data from some class.

The main feature of our method is that we use rather simple information about perturbations. Let the energy integral for (1.1)-(1.6) be valid:

$$E(t) = T(t) + \Pi_g(t) + \Pi_\alpha(t) = E(0), \tag{1.8}$$

$$2T \equiv \sum_{\lambda=1}^{2} \rho_\lambda \iiint\limits_{\Omega_\lambda^t} q_\lambda^2 \, dxdydz$$

$$2\Pi_g \equiv [\rho] g \iint\limits_{\Gamma_0} \eta^2 \, dxdy,$$

$$\Pi_\alpha \equiv \alpha \iint\limits_{\Gamma_0} (\sqrt{1 + |\nabla_2 \eta|^2} - 1) \, dxdy.$$

The function $\Pi(t) = \Pi_g + \Pi_\alpha$ is completely determined by the value of $\eta(x,y,t)$. Therefore, we will also use a notation $\Pi = \Pi(\eta)$.

If $\alpha = 0$ then integral (1.8) holds for any perturbation $\eta(x,y,t)$.

If $\alpha \neq 0$ then integral (1.8) is valid under special conditions on the curve of intersection of the surfaces $\Gamma^t$ and $\partial\Omega$ [3]. One of the conditions is that the curve of intersection is fixed. Other condition is that

$$\frac{\partial \eta}{\partial x} n_1 + \frac{\partial \eta}{\partial y} n_2 = 0$$

on this curve.

One more integral is the conservation of mass of each fluid:

$$\int_{\Gamma_0} \eta(x,y,t)\, dx dy = 0.$$

Now we confine ourselves by irrotational perturbations. It is possible because the outer force is potential. We consider solutions such that

$$\mathbf{u}_\lambda = \nabla \phi_\lambda \qquad \text{in } \Omega_\lambda^t. \tag{1.9}$$

Then the problem (1.1)-(1.6) can be rewritten in the following way:

$$\Delta \phi_\lambda = 0 \qquad \text{in } \Omega_\lambda^t,$$

$$\frac{\partial \phi}{\partial n} = 0 \text{ on } \partial\Omega,$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y} = \frac{\partial \phi}{\partial z},$$

$$\text{(C-P)} \qquad \frac{\partial}{\partial t}[\rho\phi] = -[\rho]g\eta - \frac{1}{2}[\rho|\nabla\phi|^2] + \alpha\nabla_2 \cdot \left(\frac{\nabla_2 \eta}{\sqrt{1+\mu}}\right) + f(t) \qquad \text{on } \Gamma^t,$$

$$\phi_\lambda(x,y,z,0) = \phi_\lambda^0(x,y,z) \qquad \eta(x,y,0) = \eta^0(x,y),$$

where $f(t)$ is an arbitrary function and it can be chosen in the sequel.

The main our idea is the application of some analog of the Lyapunov functional which is used in analytical mechanics. To create the functional we utilize the following facts.

Lyapunov was the first to apply the function

$$W(t) = \sum_{i=1}^{N} q_i p_i\,,$$

for the proof of instability [2]. Here $q_i$, $p_i$ are the Hamiltonian variables and $N$ is the number of degrees of freedom.

The next fact is the Hamiltonian formulation of the problem on a wave motion of a deep ideal fluid, which was introduced by Zakharov [5]. The following correspondence was used

$$q_i \to \eta, \qquad p_i \to [\rho\Psi], \qquad \sum_i \to \int_{\Gamma_0} \cdot\, dxdy,$$

where $\Psi_\lambda(x,y,t) = \phi_\lambda(x,y,\eta(x,y,t),t)$.

Therefore, we take

$$W(t) = \int_{\Gamma_0} \eta[\rho\Psi]\, dxdy$$

as the Lyapunov functional.

Our main technical result [1] is the inequality

$$\frac{dW}{dt} \geq -2E(0). \tag{1.9}$$

If we take a perturbation with initial data

$$\mathbf{u}_\lambda^0 \equiv 0 \quad (\phi_\lambda^0 = C_1 = const), \qquad \eta_0(x,y) \not\equiv 0, \tag{1.10}$$

where

$$\Pi(\eta_0) < 0, \qquad \int_{\Gamma_0} \eta_0(x,y)\, dxdy = 0,$$

then we have

$$E(0) < 0, \qquad W(0) = 0.$$

We would like to note that, if parameters $g$, $\alpha$ and $[\rho]$ satisfy some special relation ( see Remark 3.) , we can find an arbitrary small function $\eta_0$ such that $\Pi(\eta_0) < 0$.

Integrating inequality (1.9) with respect to $t$ and using the Cauchy inequality, we have

$$\int_{\Gamma_0} [\rho\Psi]^2\, dxdy + \int_{\Gamma_0} \eta^2\, dxdy \geq W(t) \geq 2|E(0)|t. \tag{1.11}$$

We have thus proved linear in time growth of quadratic norms of the Hamilton-Zakharov variables and we have the instability in this sense.

It is clear that inequality (1.11) can be rewritten in the following way

$$\sum_{\lambda=1}^{2} \rho_\lambda \int_{\Gamma^t} \phi^2\, dS + |\Pi_g(t)| \geq C_2 t. \tag{1.12}$$

Now we would like to give some interpretation of our result.
For simplicity, we consider the case where the second fluid is absent ($\rho_2 = 0,\ \rho_1 = \rho$ ).
We are able to prove

**Proposition 1.**

For each perturbation with initial data (1.10) and for any $\epsilon > 0$ there exists $t^* > 0$ such that for $t > t^*$

$$\max_{x,y} \eta(x,y,t) > H_1 - \epsilon.$$

*Proof.* Suppose the proposition is not valid.
Then

$$\int\limits_{\Gamma_0} \eta^2 dxdy \le \max\left\{(H_1-\epsilon)^2,\, H_2^2\right\} \int\limits_{\Gamma_0} dxdy = C_3.$$

and

$$\int\limits_{\Omega_1^t} |\nabla\phi|^2 dxdydz \le \frac{2}{\rho}\left(E(0) + |\Pi_g(t)|\right) \le C_4.$$

We can also choose such a function $f(t)$ in (C-P) that

$$\frac{d}{dt}\int\limits_{\Gamma_0} \rho\Psi\, dxdy = 0 \qquad \text{and} \qquad \int\limits_{\Gamma_0} \Psi\, dxdy = C_1 \int\limits_{\Gamma_0} dxdy = C_5.$$

Now we try to obtain some contradiction. With the help of the Cauchy formula we have

$$\phi(x,y,H_1-\epsilon,t) - \Psi(x,y,t) = \int\limits_{\eta(x,y,t)}^{H_1-\epsilon} \frac{\partial\phi}{\partial z}(x,y,z,t)\, dz.$$

Hence

$$\int\limits_{\Gamma_0} \phi|_{z=H_1-\epsilon}\, dxdy \le \int\limits_{\Gamma_0} \Psi dxdy + \int\limits_{\Omega_1^t} |\nabla\phi|^2 dxdydz + C_6 \le C_7,$$

and

$$\text{(1.13)} \quad \int\limits_{\Gamma_0} \Psi^2 dxdy \le 2\int\limits_{\Gamma_0} \phi^2\Big|_{z=H_1-\epsilon} dxdy + C_8 \int\limits_{\Omega_1^t} |\nabla\phi|^2 dxdydz$$

$$\le 2\int\limits_{\Gamma_0} \phi^2\Big|_{z=H_1-\epsilon} dxdy + C_9,$$

Using a simple embedding inequality for domain $\Omega_\epsilon = \Omega \cap \{z > H_1 - \epsilon\}$, we have

$$\int\limits_{\Gamma_0} \phi^2\big|_{z=H_1-\epsilon} dxdy \le C(\epsilon)\Big\{\Big[\int\limits_{\Gamma_0} \phi\big|_{z=H_1-\epsilon} dxdy\Big]^2 + \int\limits_{\Omega_\epsilon} |\nabla\phi|^2 dxdydz \le C_1(\epsilon).$$

Then inequality (1.13) gives

$$\int\limits_{\Gamma_0} \Psi^2 dxdy \le 2C_1(\epsilon) + C_9. \tag{1.14}$$

The main inequality (1.11) implies

$$\int\limits_{\Gamma_0} \Psi^2 dxdy \ge 2\rho^{-2}|E(0)|t - \rho^{-2}C_3.$$

It is quite clear that we have a contradiction with (1.14) for any

$$t > t_1 = \frac{C_3 + \rho^2(2C_1(\epsilon) + C_9)}{2|E(0)|}.$$

The proposition has been proved with $t^* = t_1$.

*Remark 1.* If the free interface touches one of the end-walls $z = H_1$, $z = -H_2$, at a finite time, then we do not have a solution over the infinite time interval. But, anyway, we have a "growing" perturbation.

*Remark 2.* Using the definition of a stable equilibrium state of a hydrodynamic system with a free surface, which was introduced by Samsonov [5],[3, p.122] we can give the definition of unstable hydrostatic equilibrium.

Let $l(\Gamma_0, \Gamma^t) = \max\limits_{x,y} |\eta(x,y,t)|$ and $\Delta(\Gamma_0, \Gamma^t)$ be the maximal angle between a tangent plane to $\Gamma^t$ and $\Gamma_0$.

**Definition.**

The hydrostatic equilibrium (1.7) is unstable if we can choose positive $L_1$, $L_2 < \dfrac{\pi}{2}$, $L_3$ such that for any positive $\lambda_1$, $\lambda_2 < \dfrac{\pi}{2}$, $\lambda_3$ there exist a perturbed initial interface $\Gamma^0 : \ z = \eta_0(x,y)$ and initial velocity field $\mathbf{u}^0_\lambda$ such that

$$l(\Gamma_0, \Gamma^0) < \lambda_1, \qquad \Delta(\Gamma_0, \Gamma^0) < \lambda_2, \qquad |\mathbf{u}^0_\lambda| < \lambda_3,$$

but there exists such an instant of time $t^* > 0$ that one of the following inequalities is valid

$$l(\Gamma_0, \Gamma^{t^*}) \ge L_1, \qquad \Delta(\Gamma_0, \Gamma^{t^*}) \ge L_2, \qquad T(t) \ge L_3.$$

Proposition 1 insures that, for the case $\rho_2 = 0$, we have proved the instability according to this definition.

*Remark 3.* If we have a special relation for $\alpha$ , $[\rho]$ and $g$ then for any $\epsilon > 0$ there exist interface $z = \eta^*(x,y)$ such that

$$\Pi(\eta^*(x,y)) < 0 \quad \text{and} \quad \max_{(x,y)\in\Gamma_0} |\eta^*(x,y)| < \epsilon.$$

In this case the main inequality gives the instability of the solution (1.7).

Let us define more exactly the relation for $\alpha$ , $[\rho]$ and $g$ . Using the Poincare inequality for sufficiently smooth function $\eta(x,y)$ such that

$$\iint\limits_{\Gamma_0} \eta(x,y)\, dxdy = 0$$

we have

$$\iint\limits_{\Gamma_0} \eta^2\, dxdy \le C \iint\limits_{\Gamma_0} |\nabla_2 \eta|^2\, dxdy,$$

where $C$ is a positive constant depending only on $\Gamma_0$. Usually we can estimate the constant $C$ from below. That is we can find a constant $\tilde{C}$ such that for any $\delta > 0$ there exist a function $\tilde{\eta}$ such that

$$\iint\limits_{\Gamma_0} \tilde{\eta}^2\, dxdy > (\tilde{C} - \delta) \iint\limits_{\Gamma_0} |\nabla_2 \tilde{\eta}|^2\, dxdy.$$

Hence

$$\iint\limits_{\Gamma_0} \left(\sqrt{1 + |\nabla_2 \tilde{\eta}|^2} - 1\right) dxdy \le \iint\limits_{\Gamma_0} |\nabla_2 \tilde{\eta}|^2\, dxdy <$$

$$< (\tilde{C} - \delta)^{-1} \iint\limits_{\Gamma_0} \tilde{\eta}^2\, dxdy,$$

and

$$\Pi(\tilde{\eta}) \le \left([\rho]g + \frac{\alpha}{\tilde{C} - \delta}\right) \iint\limits_{\Gamma_0} \tilde{\eta}^2\, dxdy < 0,$$

if

$$\left([\rho]g + \frac{\alpha}{\tilde{C} - \delta}\right) < 0.$$

Let

$$\max_{(x,y)\in\Gamma_0} \tilde{\eta} = M.$$

Then the function $\eta^*$ is taken in the form

$$\eta^* = \frac{\tilde{\eta}\varepsilon}{M}.$$

Hence, the inequality

$$[\rho]g + \frac{\alpha}{\tilde{C}} < 0$$

defines the desired condition.

## 2. The solid body rotation of an ideal fluid.

In this section we deal with two-dimensional motion of non-gravity ideal homogeneous fluid occupied the domain $\Omega^t$. The boundary of $\Omega^t$ is composed of two disjoint components — $\Gamma_1$ and $\Gamma_2^t$. $\Gamma_1$ is a circle with fixed radius and $\Gamma_2^t$ is a free surface given by the equation:

$$F(r, \theta, t) = r - \eta(\theta, t) = 0.$$

We use in this section the polar $r$, $\theta$ and the Cartesian $x = r\cos\theta$, $y = r\sin\theta$ coordinates. We assume that the outer pressure and the surface tension are absent.

In the system of frame rotating with angular velocity $\mathbf{W} = (0, 0, \omega)$ the equations of motion and the boundary conditions take the form:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \mathbf{s} = -\frac{1}{\rho}\,\nabla p + \nabla\left(\frac{1}{2}\omega^2 r^2\right), \tag{2.1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{in } D = \left\{(\mathbf{x}, t) : \mathbf{x} = (x, y) \in \Omega^t, t > 0\right\}; \tag{2.2}$$

$$\mathbf{u} \cdot \mathbf{n}_1 = 0 \qquad \text{on } \Gamma_1; \tag{2.3}$$

$$p = 0, \tag{2.4}$$

$$\eta \frac{\partial \eta}{\partial t} = \mathbf{u} \cdot \mathbf{n}_2 \sqrt{\eta^2 + \eta_\theta^2} \qquad \text{on } \Gamma_2^t. \tag{2.5}$$

Here $\mathbf{u} = (u, v)$ is the velocity of the fluid; $p$ is the pressure ; $\mathbf{s} = (-2\omega v,\ 2\omega u)$; $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit outward normal vectors to $\Gamma_1$ and $\Gamma_2^t$.

The initial conditions have the form:

$$\mathbf{u}(x, y, 0) = \mathbf{u}_0(x, y), \qquad \eta(\theta, 0) = \eta_0(\theta). \tag{2.6}$$

The functions (2.6) are subject to the restriction (2.2), (2.3) and

$$\int_0^{2\pi} (\eta^2 - R^2)\, d\theta = 0.$$

The constant $R$ is correspondent to outer radius of $\Omega^0$ in the case when the free surface $\Gamma_2^0$ is circular.

Any solution of the initial-boundary value problem (2.1)-(2.6) describes a perturbation of the solid body rotation of the fluid:

$$\mathbf{u} \equiv 0, \qquad \eta(\theta, t) \equiv R \qquad p \equiv \frac{1}{2}\rho\omega^2 (r^2 - R^2),$$

Our goal is to prove an instability of this trivial steady solution.

We consider potential perturbations

$$\mathbf{u} = (u, v) = \nabla\phi \qquad \text{in } \Omega^t, \tag{2.7}$$

which satisfy the initial conditions:

$$\mathbf{u}|_{t=0} \equiv 0, \quad \Gamma_2^0 : \ r = \eta_0(\theta) \not\equiv R, \quad \int_0^{2\pi} (\eta_0^2 - r_2^2)\, d\theta = 0. \tag{2.8}$$

In this case the Lyapunov function is taken in the form

$$W(t) = \rho \int_0^{2\pi} \Phi(\eta^2 - R^2)\, d\theta.$$

where $\Phi(\theta, t) \equiv \phi(\eta(\theta, t), \theta, t)$.

Using a special estimate from below for the time derivative of $W(t)$ we are able to prove the following statement which implies the instability.

**Proposition 2.**

Let $1 < \dfrac{R}{r_1} < \sqrt{10} - 2$.

Then any solution of the problem (2.1)-(2.8) satisfies one of the following conditions:

i) For each $t > 0$ there exists $t^* > t$ such that

$$\max_\theta |\eta(\theta, t^*) - R| \geq R - r_1;$$

ii) There exists $t_* > 0$ such that for $t > t_*$

$$\int_0^{2\pi} \Phi^2\, d\theta + \int_0^{2\pi} (\eta^2 - R^2)^2\, d\theta \geq N_1 + N_2 t.$$

where positive constants $N_1$ and $N_2$ are determined by the solution.

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